

Penrose's quasi-local mass for asymptotically anti-de Sitter space-times

Ron Kelly

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Abstract

Penrose's quasi-local mass construction is carried through for two-surfaces at infinity in asymptotically anti-de Sitter space-times. A modification of the Witten argument is given to prove a positivity property of the resulting conserved quantities.

*This work formed part of Ron Kelly's Oxford D.Phil. thesis, and the first person pronoun refers to him. It appeared in hand-written form as 'Asymptotically anti-de Sitter space-times' in Twistor Newsletter 20 (1985) pp11-23,¹ but is appearing type-set for the first time here. Footnotes marked **PT** have been added for this version by Paul Tod, in the hope of making this work available to a wider audience.*

1 The Angular Momentum Twistor

I will first briefly review the work of Ashtekar and Magnon [1]: a space-time (\hat{M}, \hat{g}_{ab}) is said to be *asymptotically anti-de Sitter* if there exists a manifold M with boundary ∂M and metric g_{ab} and a diffeomorphism from \hat{M} to $M - \partial M$ such that

1. there is a smooth real-valued function Ω on M such that $g_{ab} = \Omega^2 \hat{g}_{ab}$ on \hat{M} ;
2. $\mathcal{I} := \partial M$ is topologically $S^2 \times \mathbb{R}$ and $\Omega = 0$ on \mathcal{I} ;
3. \hat{g}_{ab} satisfies

$$\hat{R}_{ab} - \frac{1}{2} \hat{R} \hat{g}_{ab} + \lambda \hat{g}_{ab} = -8\pi G \hat{T}_{ab} \quad (1)$$

with $\lambda < 0$ and where $\Omega^{-4} \hat{T}_a^b$ has a smooth limit on \mathcal{I} ;

4. write B_{ab} for the magnetic part of the Weyl tensor of M then $B'_{ab} := \Omega^{-1} B_{ab}$ vanishes on \mathcal{I} .

In fact in [1] it was assumed instead in item 3 that $\Omega^{-3} \hat{T}_a^b$ has a limit on \mathcal{I} ; the choice made here preserves the conservation equation.

It will be convenient to introduce the notation $\hat{=}$ to mean 'equals at \mathcal{I} ', so that for example $\Omega \hat{=} 0$ and $B'_{ab} \hat{=} 0$. Examination of the Bianchi identities shows that

- (i) with $s_a = \partial_a \Omega$ we have $s^a s_a \hat{=} \lambda/3$ so that \mathcal{I} is time-like;
- (ii) by modifying Ω one can require $\nabla_a s_b \hat{=} 0$;
- (iii) $C_{abcd} \hat{=} 0$;

¹**PT**: available at <http://people.maths.ox.ac.uk/lmason/Tn/TN1-25>.

- (iv) condition 4 above is equivalent to the vanishing of the Cotton-York tensor of \mathcal{I} , so that \mathcal{I} is conformally flat. It is also equivalent to the condition

$$D_{[a}V_{b]c} \hat{=} 0, \quad (2)$$

where D_a is the intrinsic covariant derivative of \mathcal{I} and

$$V_{ab} = \Phi_{ab} - \Lambda g_{ab} - E_{ab},$$

where in turn $\Phi_{ab} = -\frac{1}{2}(R_{ab} - \frac{1}{4}Rg_{ab})$, $\Lambda = R/24$ and E_{ab} is the electric part of the Weyl tensor.

From the discussion in [2], conditions (iii) and (iv) in the second list are sufficient to conclude that \mathcal{I} may be embedded in conformally-flat space-time with the same first and second fundamental forms, and thus that 3-surface twistors exist on \mathcal{I}^2 .

We may therefore think of \mathcal{I} as the conformal infinity of anti-de Sitter space which can be embedded in the Einstein static cylinder as the product of the time axis with the equatorial 2-sphere of the 3-sphere cross-sections. In the usual way, the conformal group of \mathcal{I} is the anti-de Sitter group $O(2,3)$ and there are ten linearly independent conformal Killing vectors of \mathcal{I}^3 . Hawking [3] has shown that condition 4 in the first list is equivalent to the assumption that gravitational radiation satisfies a reflective boundary condition at \mathcal{I} .

Following [1], given a cross-section C of \mathcal{I} and a conformal Killing vector ξ^a of \mathcal{I} one defines a conserved quantity

$$Q_\xi[C] := -\frac{1}{8\pi G} \oint E'_{ab} \xi^a dS^b, \quad (3)$$

where $E'_{ab} = \Omega^{-1}E_{ab}$. This expression is conformally invariant and has a flux $F_\xi[\Delta]$ through a region Δ of \mathcal{I} bounded by two cross-sections given by

$$F_\xi[\Delta] = \int_\Delta \left(\text{Lim}_{\Omega \rightarrow 0} \Omega^{-4} \hat{T}_a^b \right) s^a \xi_b d\Sigma. \quad (4)$$

If there is no matter near to \mathcal{I} then the flux vanishes.

The Schwarzschild-anti-de Sitter metric can be written

$$ds^2 = \left(1 - \frac{2GM}{r} + a^2 r^2 \right) dt^2 - \left(1 - \frac{2GM}{r} + a^2 r^2 \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

with $\lambda = -3a^2$. Choosing $\xi^a = \frac{1}{a} \frac{\partial}{\partial t}$, which with $\Omega = r^{-1}$ is unit at \mathcal{I} in the rescaled metric, we obtain

$$Q_\xi[C] = M$$

on any cross-section C of \mathcal{I} .

To show that the expression (3) coincides with Penrose's expression for quasi-local mass and angular momentum [4] (modified for asymptotically anti-de Sitter by the subtraction of the cosmological constant term) we first need the following Lemma:

²**PT**: Three-surface twistors are defined on a three-surface Σ with normal n^a in a space-time by taking those components of the twistor equation

$$\nabla_{AA'} \omega_B = -i \epsilon_{AB} \pi_{A'}$$

for which the derivative is projected tangent to the three-surface. Writing $D_{AB} = n_{(A}^{A'} \nabla_{B)A'}$, the 3-surface twistor equation is

$$D_{AB} \omega_C = -i \pi_{(A} \epsilon_{B)C} \text{ with } \pi_A = n_A^{A'} \pi_{A'}.$$

³**PT**: So these generate conformal symmetries of \mathcal{I} .

Lemma

Suppose Σ is a hypersurface in a conformally-flat space-time, with unit normal ζ_a and $\nabla_a \zeta_b$ zero at Σ ; then the vector field

$$\xi^a := \omega^A \zeta_B^{A'} \omega^B \quad (5)$$

is a (null) conformal Killing vector at Σ if and only if

$$\nabla_{A'}(A\omega_B) = 0 \quad (6)$$

at Σ .⁴

The proof is simply by substitution of the solution of the twistor equation (6) in a conformally-flat space-time. For a general conformal Killing vector one has a sum of terms like (5).

On \mathcal{I} , with unit normal $\zeta^a = s^a/a$, take $\xi^a = 2i\omega^A \zeta_B^{A'} \omega^B$ then with a cross-section C of \mathcal{I} with unit normal t^a tangent to \mathcal{I} we have

$$\begin{aligned} Q_\xi[C] &:= -\frac{1}{8\pi G} \oint_C E'_{ab} \xi^a dS^b = -\frac{1}{8\pi G} \oint_C 2\phi_{ABCD} \zeta_{A'}^C \zeta_{B'}^D 2i\omega^A \zeta_B^{A'} t^{BB'} dS \\ &= -\frac{i}{4\pi G} \oint_C \phi_{ABCD} \omega^A \omega^B o^C \iota^D dS, \end{aligned}$$

where $\phi_{ABCD} = \Omega^{-1} \psi_{ABCD}$ and ψ_{ABCD} is the Weyl spinor. This is now recognisable as Penrose's expression for the quasi-local mass and angular momentum [4].⁵

We shall choose our momentum and angular momentum conformal Killing vectors by embedding \mathcal{I} as the boundary of anti-de Sitter space in the Einstein cylinder and then restricting solutions of the twistor equation in conformally-flat space-time to it. This process depends on how we embed Minkowski space-time into the Einstein static cylinder with respect to anti-de Sitter space-time. We choose to do this symmetrically⁶.

With respect to a constant spinor basis (α^A, β^A) in Minkowski space the solution to the twistor equation (see eg [5]) is given by

$$\omega^A = \Omega^A - ix^{AA'} \pi_{A'},$$

where

$$\Omega^A = \Omega^0 \alpha^A + \Omega^1 \beta^A, \quad \pi_{A'} = \pi^{0'} \bar{\alpha}_{A'} + \pi^{1'} \bar{\beta}_{A'}$$

and $\Omega^0, \Omega^1, \pi^{0'}, \pi^{1'}$ are complex constants. Also $x^{AA'}$ is the position vector in Minkowski space in Cartesians.

Take the Einstein cylinder to have metric

$$ds^2 = dt^2 - dr^2 - \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2),$$

⁴**PT**: The field ω^A here is a 3-surface twistor on \mathcal{I} so (6) might better be written as $D_{(AB}\omega_C) = 0$

⁵**PT**: Here there has been a calculation, that $\zeta_{A'}^C \zeta_{B'}^D \zeta_B^{A'} t^{BB'} = o^{(C} \iota^{D)}$, where o^A, ι^A are spinors representing the out and ingoing null normals to the cross-section C , normalised to have $o_A \iota^A = 1$. This formula establishes the connection between Penrose's quasi-local kinematic quantities and the Ashtekar-Magnon charges.

⁶**PT**: This means the following: take the Minkowski metric to be

$$g_M = dT^2 - dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

then this is conformal to the metric of the Einstein static cylinder written as

$$g_E = dt^2 - dr^2 - \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2),$$

with $T \pm R = \tan(t \pm r)/2$ so that the worldlines $r = 0$ and $R = 0$ coincide. The anti-de Sitter metric with $a = 1$ can then be written

$$g_{adS} = (\sec^2 r) g_E,$$

and occupies half the Einstein static cylinder, with $r \leq \pi/2$.

and introduce the null tetrad

$$\ell = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right), \quad n = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r} \right), \quad m = \frac{1}{\sqrt{2} \sin r} \left(\frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right).$$

This tetrad implicitly defines a spinor dyad (o^A, ι^A) in the usual way and expanded in this dyad the solution of the twistor equation is

$$\omega^A = \omega^0 o^A + \omega^1 \iota^A$$

with

$$\begin{aligned} \omega^0 &= \sqrt{2} \cos\left(\frac{1}{2}(t+r)\right) \left(\Omega^0 e^{-i\phi/2} \cos(\theta/2) + \Omega^1 e^{i\phi/2} \sin(\theta/2) \right) \\ &\quad + i \sin\left(\frac{1}{2}(t+r)\right) \left(-\pi^{0'} e^{i\phi/2} \sin(\theta/2) + \pi^{1'} e^{-i\phi/2} \cos(\theta/2) \right), \\ \omega^1 &= \sqrt{2} \cos\left(\frac{1}{2}(t-r)\right) \left(-\Omega^0 e^{-i\phi/2} \sin(\theta/2) + \Omega^1 e^{i\phi/2} \cos(\theta/2) \right) \\ &\quad + i \sin\left(\frac{1}{2}(t-r)\right) \left(\pi^{0'} e^{i\phi/2} \cos(\theta/2) + \pi^{1'} e^{-i\phi/2} \sin(\theta/2) \right). \end{aligned} \quad (7)$$

Since the twistor equation is conformally invariant, this is also a solution of the twistor equation on the Einstein cylinder.

From these expressions we may calculate $Q_\xi[C]$ with the conformal Killing vector

$$\xi^a = 2i \zeta^{A'}_B \omega^{(A} \tilde{\omega}^{B)}$$

and the cross-section C in $r = \pi/2$ as

$$Q_\xi[C] = -\frac{i}{4\pi G} \oint_C \phi_{ABCD} \omega^A \tilde{\omega}^B dS^{CD} = A_{\alpha\beta} Z^\alpha \tilde{Z}^\beta, \quad (8)$$

with

$$Z^\alpha = (\Omega^A, \pi_{A'}), \quad \tilde{Z}^\alpha = (\tilde{\Omega}^A, \tilde{\pi}_{A'})$$

and

$$A_{\alpha\beta} = \begin{pmatrix} 2\Phi_{AB} & P_{A'}^{B'} \\ P_B^{A'} & \Phi_{A'B'} \end{pmatrix}$$

In particular the 4 components of momentum $P^{\mathbf{a}}$ are given by

$$Q_\xi[C] := -\frac{1}{8\pi G} \oint E'_{ab} \xi^a dS^b$$

with ξ^a one of $(\gamma^a, \frac{1}{2}(\eta^a + \bar{\eta}^a), \frac{1}{2i}(\eta^a - \bar{\eta}^a), \beta^a)$ where

$$\gamma = \frac{\partial}{\partial t}, \quad \eta = e^{i\phi} \left(\sin t \sin \theta \frac{\partial}{\partial t} - \cos t \cos \theta \frac{\partial}{\partial \theta} - i \cos t \frac{\partial}{\partial \phi} \right),$$

$$\beta = \cos \theta \sin t \frac{\partial}{\partial t} + \sin \theta \cos t \frac{\partial}{\partial \theta}.$$

For the case of Schwarzschild–anti-de Sitter $P^{\mathbf{a}} = (M, 0, 0, 0)$ and $\Phi_{AB} = 0$ and we recover M as the mass.

The angular momentum automatically obeys the Hermiticity property

$$A_{\alpha\beta} I^{\beta\gamma} = \overline{A_{\gamma\beta} I^{\beta\alpha}} \quad (9)$$

with respect to the infinity twistor⁷

$$I^{\alpha\beta} = \begin{pmatrix} -\frac{1}{2}\epsilon^{AB} & 0 \\ 0 & \epsilon_{A'B'} \end{pmatrix}. \quad (10)$$

⁷**PT**: Remember that, in this part of the discussion, $a = 1$; otherwise a^2 appears in (10).

2 A positive energy theorem

We first need to look at the work of Gibbons et al [6]. Consider a space-like hypersurface Σ in an asymptotically anti-de Sitter space-time (possibly with an inner boundary on a past or future apparent horizon \mathcal{H}) which asymptotically approaches the $t = 0$ cross-section of \mathcal{I} . Define a *supercovariant* derivative on a 4-spinor $(\alpha^A, \beta^{A'})$ by

$$\begin{aligned}\hat{\nabla}_{MM'}\alpha_A &= \nabla_{MM'}\alpha_A + \frac{a}{\sqrt{2}}\epsilon_{MA}\beta_{M'} \\ \hat{\nabla}_{MM'}\beta_{A'} &= \nabla_{MM'}\beta_{A'} + \frac{a}{\sqrt{2}}\epsilon_{M'A'}\alpha_M.\end{aligned}\tag{11}$$

Now suppose that D_a is the projection into Σ of ∇_a and introduce the *supercovariant Witten equation* on Σ :

$$\begin{aligned}\hat{D}_{AA'}\alpha^A &= D_{AA'}\alpha^A + \frac{3a}{2\sqrt{2}}\beta_{A'} = 0 \\ \hat{D}_{AA'}\beta^{A'} &= D_{AA'}\beta^{A'} + \frac{3a}{2\sqrt{2}}\alpha_A = 0.\end{aligned}$$

Then one can show that

$$-D_m \left(t^{AA'} (\bar{\alpha}_{A'} \hat{D}^m + \bar{\beta}_A \hat{D}^m \beta_{A'}) \right) = -t^{AB'} (\hat{D}_m \alpha_A \overline{\hat{D}^m \alpha_B} + \hat{D}_m \beta_{B'} \overline{\hat{D}^m \beta_{A'}}) + 4\pi G T_{ab} t^a \xi^b, \tag{12}$$

where $\xi^a = \alpha^A \bar{\alpha}^{A'} + \beta^{A'} \bar{\beta}^A$. Thus if T_{ab} satisfies the Dominant Energy Condition, then the RHS is non-negative. We may choose the spinors $\alpha^A, \beta^{A'}$ to obey boundary conditions on an inner apparent horizon \mathcal{H} such that

$$t^{AA'} (\bar{\alpha}'_A \hat{D}^m \alpha_A + \bar{\beta}_A \hat{D}^m \beta_{A'}) N_m = 0 \text{ at } \mathcal{H},$$

where N_m is the normal to \mathcal{H} lying in Σ .

If we use Green's Theorem on the identity (12) we therefore obtain

$$-\oint_{t=0} t^{AA'} (\bar{\alpha}'_A \hat{D}^m \alpha_A + \bar{\beta}_A \hat{D}^m \beta_{A'}) s_m dS \geq 0. \tag{13}$$

This integral is finite providing that

$$\hat{D}_m \alpha_A \rightarrow 0, \quad \hat{D}_m \beta_{A'} \rightarrow 0 \text{ in the limit at } \mathcal{I}.$$

In [6] it was proposed that the boundary term could be written as

$$4\pi G \left(P^{(5)AA'} (\overset{\circ}{\alpha}_A \bar{\overset{\circ}{\alpha}}_{A'} + \overset{\circ}{\beta}_{A'} \bar{\overset{\circ}{\beta}}_A) - \lambda^{AB} \overset{\circ}{\alpha}_A \bar{\overset{\circ}{\beta}}^B - \bar{\lambda}^{A'B'} \bar{\overset{\circ}{\alpha}}_{A'} \overset{\circ}{\beta}_{B'} \right) \tag{14}$$

(credited as a private communication from D.Z.Freedman) where $\overset{\circ}{\alpha}_A, \overset{\circ}{\beta}_{A'}$ are the limits on \mathcal{I} of the supercovariantly constant spinors $\alpha_A, \beta_{A'}$ (after division by a suitable power of the conformal factor).

For Schwarzschild–anti-de Sitter the component $P^{(5)0} := \frac{1}{\sqrt{2}}(P^{(5)00'} + P^{(5)11'})$ is a (constant, positive) multiple of the mass parameter M . In general, the inequality (13) implies $P^{(5)0} \geq 0$ so that, if we identify this term with the mass, as was done in [6], then we have shown that the mass is positive. But, as shall be described, if we evaluate the integral in (13) explicitly we obtain Penrose's expression $Q_\xi[C]$ and then we find that (13) contains much more information about $A_{\alpha\beta}$.

We first write the metric in terms of coordinates (u, s, θ, ϕ) as⁸

$$g = \frac{1}{\sin^2 s} (du^2 - \frac{2}{a} du ds - \cos^2 s (d\theta^2 + \sin^2 \theta d\phi^2)) + O(s^{-1}) \quad (15)$$

where the $O(s^{-1})$ terms do not contain ds^2 . Here the constant u surfaces are outgoing null hypersurfaces meeting \mathcal{I} which is located at $s = 0$, $(x^2, x^3) = (\theta, \phi)$ label the null generators of the constant u surfaces, and s is a parameter (not affine) on each null generator. We shall choose a null tetrad as follows:

$$\ell_a dx^a = \frac{du}{\sin s}, \text{ so that } \ell^a \partial_a = -a \sin s \frac{\partial}{\partial s},$$

n^a is the ingoing null normal to the 2-spheres of constant u and s , and m^a is a complex null tangent to these 2-spheres, chosen so that the NP spin coefficient ϵ is real⁹. We thus have

$$\ell^a \partial_a = -a \sin s \frac{\partial}{\partial s}, \quad (16)$$

$$n^a \partial_a = \sin s \left(\frac{\partial}{\partial u} + aU \frac{\partial}{\partial s} + X^k \frac{\partial}{\partial x^k} \right), \quad (17)$$

$$m^a \partial_a = \frac{a}{\sqrt{2}} \xi^k \frac{\partial}{\partial x^k}, \quad (18)$$

for real U, X^k and complex ξ^k . For anti-de Sitter space-time

$$U = \frac{1}{2}, \quad X^k = 0, \quad \xi^2 = \tan s, \quad \xi^3 = -i \tan s / \sin \theta,$$

so that these are the values at \mathcal{I} in general. Also from the assumption that $\Omega^{-4} \hat{T}_{ab}$ is finite at \mathcal{I} we have the asymptotic behaviour of the NP curvature quantities:

$$\Phi_{ij} = O(s^4), \quad \Lambda = -\lambda/6 = O(s^4), \quad \psi_i = O(s^3).$$

We may therefore solve for the NP spin coefficients as power series expansions in s and then we may calculate $\alpha_A, \beta_{A'}$ similarly. We find

$$\alpha^A = s^{-1/2} \sum_0^\infty \alpha_i^A s^i, \quad \beta^{A'} = s^{-1/2} \sum_0^\infty \beta_i^{A'} s^i,$$

with $\beta_0^{A'} = \sqrt{2} \alpha_0^A s_A^{A'}$ (this equality is an identity in the rescaled space-time since $s^a \hat{=} \frac{1}{2} \ell^a - n^a$).

The integral on the LHS of (13) becomes

$$\begin{aligned} & -\frac{\sqrt{2}}{a^3} \oint_{t=0, s=0} \left(\psi_1^{(3)} \alpha_0^0 + \psi_2^{(3)} (\alpha_0^1 \bar{\beta}_0^0 + \alpha_0^0 \bar{\beta}_0^1) + \psi_3^{(3)} \alpha_0^1 \bar{\beta}_0^1 \right) dS_0 \\ & = -\frac{i}{a^3} \oint_{t=0, s=0} \left(\psi_1^{(3)} \omega^0 \tilde{\omega}^0 + \psi_2^{(3)} (\omega^1 \tilde{\omega}^0 + \omega^0 \tilde{\omega}^1) + \psi_3^{(3)} \omega^1 \tilde{\omega}^1 \right) dS_0 \\ & = \frac{4\pi G}{a^3} A_{\alpha\beta} Z^\alpha \tilde{Z}^\beta, \end{aligned}$$

where $\psi_i^{(3)}$ means the $O(s^3)$ term in ψ_i , and $\omega^A \hat{=} \alpha_0^A, \tilde{\omega}^A \hat{=} -i\sqrt{2}\bar{\beta}_0^A$ which are found to be 2-surface twistors on the $t = 0$ cross-section of \mathcal{I} . But in (7) we have expressions for ω^0, ω^1 in terms of $(\Omega^A, \pi_{A'})$ at $t = 0$, and the relation between α_0^A and $\bar{\beta}_0^{A'}$ gives us that

$$\tilde{Z}_\alpha = 2I^{\alpha\beta} \bar{Z}_\beta,$$

⁸**PT**: The leading terms in this expression correspond to the metric g_{adS} of footnote 6 but with general a .

⁹**PT**: This can be accomplished by rotating m^a in the 2-plane tangent to the chosen 2-spheres.

using (10). We have therefore shown, by (13), that

$$A_{\alpha\beta} I^{\beta\gamma} Z^\alpha \bar{Z}_\gamma \geq 0. \quad (19)$$

In particular this implies that P^a is time-like and future-pointing¹⁰. From the angular momentum twistor $A_{\alpha\beta}$ we may calculate the associated mass m_P as

$$m_P^2 = -\frac{1}{2} A_{\alpha\beta} \bar{A}^{\alpha\beta} = P_a P^a - \Phi_{AB} \Phi^{AB} - \bar{\Phi}_{A'B'} \bar{\Phi}^{A'B'}.$$

Inequality (19) implies that m_P^2 is non-negative and also provides a further inequality relating components of P^a and Φ_{AB} ¹¹. An alternative definition of mass is

$$m_D^4 = 4 \det A_{\alpha\beta} = 4 \epsilon^{\alpha\beta\gamma\delta} A_{\alpha 1} A_{\beta 2} A_{\gamma 3} A_{\delta 4},$$

and m_D^4 is also non-negative by virtue of (19). When $\Phi_{AB} = 0$, we have $P^a P_a = m_P^2 = m_D^2$ but in general the masses are different.

Acknowledgements

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References

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¹⁰**PT**: Kelly didn't make the point explicitly but this result also provides *rigidity* in the sense that if there exists a Z^α giving zero in (19) then the space-time is exactly anti-de Sitter at least near Σ . This follows from (12): Dominant Energy implies that T_{ab} vanishes at Σ , and therefore everywhere; and one can evolve $(\alpha^A, \beta^{A'})$ from Σ to obtain a solution of (11) in the space-time, which forces the Weyl tensor to vanish.

¹¹**PT**: In this connection, see [7].